On the Thermodynamics of Gödel Black Holes

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After a brief review of Gödel-type universes in string theory, we discuss some intriguing properties of black holes immersed in such backgrounds. Among these are the upper bound on the entropy that points towards a finite-dimensional Hilbert space of a holographically dual theory, and the minimum black hole temperature that is reminiscent of the Hawking-Page transition. Furthermore, we discuss several difficulties that are encountered when one tries to formulate a consistent thermodynamics of Gödel black holes, and point out how they may be circumvented.

1 Introduction

One of the most surprising results of the classification of all BPS solutions to $N = 2$ supergravity in five dimensions [1] was the existence of a maximally supersymmetric Gödel-type universe, with metric and $\mathrm{U}(1)$ gauge field given by

$$ds^2 = -(dt + jr^2\sigma_3)^2 + dr^2 + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)\,, \qquad A = \frac{\sqrt{3}}{2}jr^2\sigma_3\,, \tag{1}$$

where

$$\begin{aligned} \sigma_1 &= \sin\phi d\theta - \cos\phi\sin\theta d\psi\,, \\ \sigma_2 &= \cos\phi d\theta + \sin\phi\sin\theta d\psi\,, \\ \sigma_3 &= d\phi + \cos\theta d\psi\,. \end{aligned} \tag{2}$$

denote right-invariant one-forms on SU(2), with Euler angles (θ, ϕ, ψ). j is a real parameter responsible for the rotation of (1). This solution solves the equations of motion following from the action

$$I = \frac{1}{16\pi G}\int d^5x\sqrt{-g}[R - F^2] - \frac{1}{24\sqrt{3}\pi G}\int d^5x\,\epsilon^{\mu\nu\alpha\beta\gamma}F_{\mu\nu}F_{\alpha\beta}A_\gamma\,. \tag{3}$$

Like the original Gödel universe [2], presented by Kurt Gödel in 1949 on occasion of Albert Einstein's 70th birthday, this solution is homogeneous: It has a nine-dimensional group of bosonic isometries $\mathrm{H}(2) \ltimes (\mathrm{SU}(2)_R \times \mathrm{U}(1)_L)$ [3], where $\mathrm{H}(2)$ denotes the Heisenberg group with five generators. There are further common features of (1) and its four-dimensional cousin: The stress tensor of the Maxwell field in (1) is that of pressureless dust with energy density proportional to j^2 [1]. In addition, just like the original Gödel universe, the solution (1) suffers from closed timelike curves (CTCs); the induced metric on hypersurfaces of constant t and r becomes Lorentzian for $r > 1/j$.

One can lift the five-dimensional Gödel-type universe to $D = 11$ supergravity. The solution is then just a product spacetime of (1) and a six-dimensional flat space $\mathbb{R}^6$. This configuration is highly supersymmetric, it preserves 20 supercharges [1]. Let us denote the coordinates of $\mathbb{R}$ by y^a. Dimensional reduction along say y^6 and subsequent T-duality along y^5 yields then the type IIB pp-wave resulting from the Penrose limit of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ [3]. Actually what one gets is a compactified pp-wave, whose CTCs can be eliminated by going to the universal covering space. Note that this is not possible for the solution (1), which is already topologically trivial.

The appearance of CTCs creates of course several pathologies, for instance the possibility of time-travel or the fact that the Cauchy problem in such spacetimes is always ill-defined, to mention only two of them. There is an ongoing research activity on Gödel-type universes, with the general aim to shed light on the status of closed timelike curves in string theory. One would like to understand how string theory resolves the pathologies mentioned above, and if we should discard solutions like (1), in spite of their high amount of supersymmetry. We will not discuss these issues here, and refer the reader to [3, 4, 5, 6] and references therein. Apart from questions related to causality problems, the interest in Gödel-type spacetimes is also motivated by holography: Similar to de Sitter spaces, the Gödel universe shows observer-dependent holographic screens [3], and there is evidence for a finite-dimensional Hilbert space of a putative holographically dual theory. The Gödel-type universe represents thus a supersymmetric laboratory for addressing conceptual puzzles of de Sitter holography [3]. As we will see, of particular interest in these contexts are black holes immersed in Gödel universes, which we will discuss in the remainder of this paper.

2 Black Holes in Gödel Universes

Gimon and Hashimoto presented a solution that can be viewed as a Kerr black hole (with two equal rotation parameters $l_1 = l_2 \equiv l$) embedded in the five-dimensional Gödel universe (1)[1]. The metric and gauge field are given by [7]

$$ds^2 = -f(r)\left(dt + \frac{a(r)}{f(r)}\sigma_3\right)^2 + \frac{dr^2}{V(r)} + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2) + \frac{r^2 V(r)}{4f(r)}\sigma_3^2\,, \qquad A = \frac{\sqrt{3}}{2} j r^2 \sigma_3\,, \tag{4}$$

where

$$f(r) = 1 - \frac{2m}{r^2}\,, \qquad a(r) = jr^2 + \frac{ml}{r^2}\,, \qquad V(r) = 1 - \frac{2m}{r^2} + \frac{16j^2m^2}{r^2} + \frac{8jml}{r^2} + \frac{2ml^2}{r^4}\,. \tag{5}$$

m represents the "mass" parameter of the solution. For $j = 0$, (4) reduces to the Kerr black hole with equal rotation parameters. For $l = m = 0$, we recover the Gödel universe (1). In what follows, we shall consider only the case $l = 0$, i. e. , the Schwarzschild-Gödel black hole. This has an event horizon at $r_H = \sqrt{2m(1-8j^2m)}$ and an ergosphere at $r_{erg} = \sqrt{2m}$. For $r = r_{CTC} = \sqrt{1-8j^2m}/2j$ (the velocity of light surface), ∂_ϕ becomes lightlike. If $r > r_{CTC}$, ∂_ϕ is timelike and CTCs appear. Note that $r_H < r_{CTC}$ for $0 < m < 1/8j^2$. $m_{max} = 1/8j^2$ is the maximal mass parameter; for $m > m_{max}$ there is no horizon. Notice also that for $m \neq 0$, the electromagnetic stress tensor can no more be written in perfect fluid form.

The vector field

$$\mathcal{N}u = \partial_t + \frac{4j}{1 - 4j^2r^2 - 8j^2m}\partial_\phi \tag{6}$$

is orthogonal to the natural time slicing determined by t and becomes null on the horizon. u is the future directed normal field and the lapse

$$\mathcal{N}^2 = \frac{r^2 - 2m + 16j^2m^2}{r^2(1 - 4j^2r^2 - 8j^2m)} \tag{7}$$

[1] An extremal supersymmetric Reissner-Nordström-Gödel black hole was found in [1], and its properties were studied in [6].

vanishes on the horizon but is infinite on r_{CTC}. The horizon limit field

$$K = \partial_t + \omega_H \partial_\phi \,, \qquad \omega_H = \frac{4j}{(1-8j^2m)^2}\,, \tag{8}$$

is a timelike Killing field for $r \geq r_H$. Note that the horizon angular velocity ω_H is nonzero because we have a black hole in a rotating universe rather than a rotating black hole that lives in a static background.

It is tempting to associate to this black hole an entropy

$$S = \frac{A_H}{4G} = \frac{\pi^2}{2G}\sqrt{8m^3(1-8j^2m)^5}\,. \tag{9}$$

Figure 1 shows S as a function of the mass parameter m. We see that, in contrast to black holes in

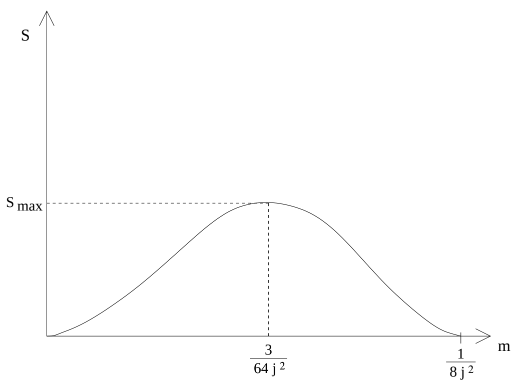

Fig. 1 Entropy S of the Schwarzschild-Gödel black hole as a function of the mass parameter.

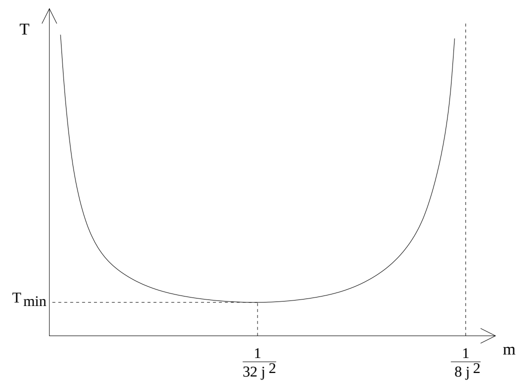

Fig. 2 Temperature T as a function of m.

asymptotically flat spaces or in AdS, Gödel black holes cannot have an arbitrarily large entropy. S becomes maximal for $m = 3/64j^2$ and decreases when m is further increased. As the Schwarzschild-Gödel black hole represents a finite temperature excitation above the supersymmetric Gödel ground state, it seems that there is an upper bound on the entropy of such excitations. Similar to the case of de Sitter gravity, whose holographic dual was argued to have a finite number of states [8] (cf. also [9] for a recent review), this seems to point towards a finite-dimensional Hilbert space of the quantum description. In de Sitter space, the entropy of a black hole is bounded from above by the entropy of the Nariai solution, which is the largest black hole that can fit within the cosmological horizon. Something similar happens for the Gödel black hole, where the surface $r = r_{CTC}$ (beyond which CTCs appear) acts effectively like a box: The three-sphere that is tangent to the horizon develops timelike directions for $r > r_{CTC}$, but timelike vectors cannot be tangent to null surfaces, so there can be no horizons for $r > r_{CTC}$. That's why there is an upper bound on the mass parameter and on the entropy of the Gödel black hole.

The horizon temperature is given in terms of the surface gravity κ,

$$T = \frac{\kappa}{2\pi} = \frac{1}{2\pi\sqrt{2m(1-8j^2m)^3}}\,. \tag{10}$$

Alternatively, it can be found by demanding smoothness of the Euclidean section, which is obtained by the analytic continuation

$$t \to \tau = -it\,, \qquad j \to J = ij\,. \tag{11}$$

This implies that the Euclidean time τ is identified modulo $\beta = 1/T$, and $\phi \sim \phi + \beta\Omega_H$, where

$$\Omega_H = \frac{4J}{(1+8J^2m)^2}\,. \tag{12}$$

Quite surprisingly, it seems that the Euclidean section is perfectly well-behaved in the whole range of r, in spite of the pathologies encountered in its Lorentzian counterpart.

Figure 2 shows T as a function of m. Obviously there is a minimum temperature $T_{min} = 16j/(3\sqrt{3}\pi)$ for $m = 1/32j^2$, so that the Schwarzschild-Gödel black hole can exist only above this temperature. This is reminiscent of Schwarzschild-AdS black holes, where a similar behaviour leads to the Hawking-Page phase transition.

In order to study more in detail the thermodynamics of the Gödel black hole, one needs a notion of mass and angular momentum in Gödel universes. It is clear that m and j are not the thermodynamical parameters, because the partial derivative of the entropy (9) with respect to m, keeping j fixed, does not yield the correct inverse temperature. One possible way to define a mass is by means of the Euclidean action. It is straightforward to show that in our case, the on-shell Euclidean action reduces to the boundary term

$$I_E = \frac{1}{12\pi G}\int_{\partial\mathcal{M}} d^4S_\mu A_\nu F^{\mu\nu} - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^4x\sqrt{h}K\,, \tag{13}$$

where $d^4S_\mu = n_\mu d^4x$ with n_μ the outward pointing unit normal to the boundary $\partial\mathcal{M}$, $h_{\mu\nu}$ is the induced metric on the boundary, and K denotes the trace of its extrinsic curvature. The last term is the Gibbons-Hawking boundary term necessary in order to have a well-defined variational principle. Taking $\partial\mathcal{M}$ to be a surface of constant r, one finds

$$I_E = \frac{\pi\beta}{32G}(3r^2 + 8J^2mr^2 - 4m - 32J^2m^2 - 4J^2r^4)\,. \tag{14}$$

We were not able to regularize the divergence in I_E for $r \to \infty$ by subtracting a suitable background. The obvious background would be the solution with $m = 0$, i. e. , the supersymmetric Gödel universe. However, strictly speaking, the Gödel black hole does not approach the geometry with $m = 0$ asymptotically, because

$$g_{\mu\nu} - g_{\mu\nu}^{m=0} \sim -2j^2mr^2\sigma_3^2 \qquad \text{for} \quad r \to \infty\,. \tag{15}$$

The absence of a proper background to which the spacetime asymptotes is also one of the reasons that the usual Abbott-Deser construction to compute the mass and angular momentum does not work here. (Another reason is the fact that the conserved charges calculated with the Abbott-Deser construction involve surface integrals at infinity, but, as the induced metric on such surfaces becomes Lorentzian due to the presence of CTCs, these integrals would be imaginary). A possible way out could be to put the Gödel black hole in a box $R < r_{CTC}$, and to compute the stress-energy-momentum content of the bounded spacetime region $r \leq R$ using the Brown-York formalism [10]. In the case of the four-dimensional Schwarzschild black hole, this yields a first law in the form [10]

$$dE = pdV + T_{local}dS\,, \tag{16}$$

where

$$T_{local} = (8\pi M\sqrt{1 - 2M/R})^{-1} \tag{17}$$

is the local Tolman temperature, blueshiftet from infinity to a finite radius R. Note the appearance of the additional term pdV in the first law, which comes from the fact that one is considering a finite volume $V = 4\pi R^2$. One can now try to generalize this to the case of the Schwarzschild-Gödel black hole. Work in this direction is in progress.

Alternatively, one can proceed as in [11] to obtain a mass formula by integrating the Killing identity

$$\nabla_\nu\nabla_\mu\xi^\nu = R_{\mu\sigma}\xi^\sigma = 2(F_{\mu\nu}F_\sigma{}^\nu - \frac{1}{6}g_{\mu\sigma}F^2)\xi^\sigma\,, \tag{18}$$

where ξ is a Killing vector, on a spacelike hypersurface Σ_t of constant t from the black hole horizon r_H to some $R < r_{CTC}$. Using $\xi = \partial_t$ we obtain for the total mass M contained in the region $r \leq R$ the Smarr-like formula

$$M = \frac{3}{16\pi G}\kappa A_H + \frac{3}{2}\omega_H J_H + \frac{3}{2}\int_{\Sigma_t}(T_{\mu\sigma} - \frac{1}{3}Tg_{\mu\sigma})\xi^\sigma d\Sigma^\mu \,, \tag{19}$$

where

$$J_H = -\frac{1}{16\pi G}\int_{\mathrm{Hor}} \nabla_\mu \tilde{K}_\nu d\Sigma^{\mu\nu} \,, \qquad \tilde{K} = \partial_\phi \,, \tag{20}$$

is the angular momentum of the horizon, and

$$T_{\mu\sigma} = \frac{1}{4\pi G}(F_{\mu\nu}F_\sigma{}^\nu - \frac{1}{4}g_{\mu\sigma}F^2) \tag{21}$$

denotes the stress tensor of the electromagnetic field. The last term in (19) represents the contribution of the gauge field to the energy in the region $r \leq R$. Evaluating (19) yields

$$M = \frac{3\pi}{4G}[m(1 - 8j^2 m) + 2j^2 R^2(R^2 - 2m)] \,. \tag{22}$$

Further difficulties are related to the different notions of charge that appear in presence of Chern-Simons terms [12]. To see this, consider the Maxwell equations

$$\nabla_\mu F^{\mu\nu} = 4\pi J^\nu_{ext} \,, \tag{23}$$

where J_{ext} denotes an external source. Integrating this on a spacelike hypersurface Σ and using Gauss' law yields

$$Q_M = \frac{1}{4\pi}\int_{\partial\Sigma} F^{\mu\nu} d\Sigma_{\mu\nu} \tag{24}$$

for the "Maxwell charge" $Q_M = \int_\Sigma J^\nu_{ext} d\Sigma_\nu$. (24) diverges for the Gödel black hole, if $\partial\Sigma$ is a surface of constant t, r and $r \to \infty$.

In our case, we have a Chern-Simons term, and thus the Maxwell equations are modified to

$$\nabla_\mu (F^{\mu\nu} + J^{\mu\nu}) = 4\pi J^\nu_{ext} \,, \tag{25}$$

with the Chern-Simons current

$$J^{\mu\nu} = \frac{1}{\sqrt{3}}\epsilon^{\mu\nu\alpha\beta\gamma} A_\alpha F_{\beta\gamma} \,. \tag{26}$$

Proceeding as before yields the "Page charge"

$$Q_P = \frac{1}{4\pi}\int_{\partial\Sigma}(F^{\mu\nu} + J^{\mu\nu}) d\Sigma_{\mu\nu} \,, \tag{27}$$

which vanishes for the Gödel black hole. These different notions of charge raise the question which one enters the first law of black hole mechanics.

If we use the Page charge, which takes into account only external sources, but not the contributions that come from the Chern-Simons current $J^{\mu\nu}$, the Schwarzschild-Gödel solution is not electrically charged. It might possess, however, a magnetic dipole moment μ. Something similar happens for supertubes, to

which the Gödel spacetimes are closely related [5]. For instance the type IIA supertube carries no D2-brane charge, but it does have a D2-dipole moment [13]. The question is thus if there appears a term $Bd\mu$ in the first law of the Gödel black hole (B denotes the magnetic field). As far as we are aware, up to now there are no known examples of black holes that have such a term in the first law[2]. To determine μ is nontrivial; the magnetic dipole moment cannot be simply read off from the asymptotic behaviour of the electromagnetic field, because the solution is not asymptotically flat.

Adopting the notion of Page charge and assuming the absence of a magnetic dipole moment, the first law for the Gödel black hole should have the form

$$dE = TdS + \omega_H dJ \,. \tag{28}$$

This can be integrated to get expressions for E and J. For dimensional reasons it is clear that E and J must have the form

$$E = \frac{m}{G} f(x)\,, \qquad J = \frac{jm^2}{G} h(x)\,, \tag{29}$$

where f, h are functions of $x \equiv j^2 m$ only. The first law (28) implies then

$$\begin{aligned} E &= \frac{3\pi m}{4Gx}(1-8x)\left[\frac{1}{24} + 4C(1-8x)^3\right]\,, \\ J &= \frac{\pi j m^2}{32Gx^2}(1-8x)^3\left[\frac{1}{6} - 4x + 16C(1-8x)^3\right]\,, \end{aligned} \tag{30}$$

with C denoting an integration constant. Notice that it is not possible to choose C such that E coincides with the total mass within the velocity of light surface $r = r_{CTC}$, obtained by choosing $R = r_{CTC}$ in the Smarr formula (19), (22). Furthermore, for no choice of C does E reduce to the total mass within the holographic Bousso screen (located at $r = \sqrt{3(1-8j^2m)}/4j$ [7]), or to the expression $3\kappa A_H/(16\pi G) + 3\omega_H J_H/2$ in (19), that represents somehow the mass of the black hole (being the total mass minus the contribution of the gauge field). It is easy to see that also $J \neq J_H$ for all values of C. This means that the mass E and the angular momentum J that enter the first law must also have contributions from the electromagnetic field. Requiring E to reduce to its correct value $E = 3\pi m/4G$ in the limit $j \to 0$ fixes $C = -1/96$.

3 Final Remarks

We saw that black holes in Gödel universes have some interesting properties, like the upper bound on the entropy, that points towards a finite-dimensional Hilbert space of the quantum description, or the minimum temperature that is reminiscent of Schwarzschild-AdS black holes. For the latter, the minimum temperature leads to the Hawking-Page phase transition, but in the Gödel case it is difficult to imagine something similar, because a phase transition cannot occur in systems with a finite number of degrees of freedom.

Perhaps a detailed analysis of the thermodynamical behaviour of Gödel black holes can shed more light on these questions. As we saw, the formulation of a consistent thermodynamics is hindered by the difficulties that one encounters in defining conserved charges like mass and angular momentum. These problems are intimately related to the appearance of CTCs.

One way to overcome them may be to embed the Gödel black hole in an AdS background[3] and to study the thermodynamics of this ”Schwarzschild-Gödel-AdS black hole”. The motivation for this rests on the fact that a negative cosmological constant Λ can act as a regulator for CTCs: In the Gödel-AdS solution

[2] The black rings of [14] carry a dipole moment that is independent of their electric charge, but the thermodynamics of these objects has not yet been worked out.

[3] Unfortunately the corresponding metric is not yet known.

found in [15], all CTCs disappear if the absolute value of Λ is large enough compared to the Gödel parameter j [16]. The difficulties in defining conserved charges and formulating a consistent thermodynamics, present in the case $\Lambda = 0$ and discussed in detail above, might then disappear. Having determined the thermodynamic potentials, one can then study the limit when the cosmological constant approaches its critical value where CTCs show up, and see whether at that point phase transitions appear in the system. We leave an analysis of these issues for a future publication.

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